Cold Atoms and Holography

The Gravity Dual of the Schrödinger Equation

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May 5, 2019

1 Prologue

In the late 1990's, both cold-atom physics and string theory independently experienced revolutions due to seminal discoeries in each field. On the ultracold frontier, the first Bose-Einstein condensate (BEC)—a macroscopic quantum wave formed by the "condensation" of the atoms in a very cold gas of bosons—was synthesized in 1995. Two years later, Juan Maldacena took the theoretical physics community by storm by proposing a surprising correspondence between a particular type of string theory and a model of quantum field theory. Both fields took off in their own directions and developed very different paradigms for thinking about nature; nevertheless, we now present an interesting attempt to bring them together.

2 The Unitary Fermi Gas

Prof. David Weld of UCSB put it best when he outlined the motivations for studying ultracold quantum gases: he described the field of atomic, molecular, and optical (AMO) physics as "experimental condensed-matter theory." What he meant was that AMO systems are pristine environments in which one can model and simulate phenomena that hardcore condensed-matter theorists could hitherto only dream of. Ultracold gases, by virtue of being cold, slow down quantum dynamics to sizes and timescales observable by humans, often with the naked eye. They allow detailed study of many-body physics, out-of-equilibrium phenomena, precision measurement and atomic clocks, physics beyond the standard model such as CP violation and tests of general relativity, quantum simulation, computing, and information, and much more. Among these topics are strongly correlated systems, which defy perturbative methods—the assumption that physics = (trivial) + ε(nontrivial) + $O(\varepsilon^2)$ for small ε—and are therefore of great interest to theorists and experimentalists alike.

2.1 Quantum Gases at Unitarity

In the context of quantum gases, the phrase "strongly correlated" means that every atom in the gas has a non-negligible effect on every other atom. For instance, a large cloud of charged particles is weakly interacting: even though the Coulomb repulsion between particles is strong, it falls off as the square of the distance between them, so particles at the edges of the cloud do not affect each other as much as nearby ones. We can formalize what we mean by the range of interactions in a gas by the following opaque definition:

Definition 2.1. The *scattering length* a is given by $k\cot\delta(k)\big) = -\frac{1}{a}$, where $k \to 0$ is the momentum of the interacting particles and $\delta(k)$ is the shift in phase experienced by the wavefunction of the scattering particle.

This definition tells us nothing useful; instead, let us visualize the interparticle interactions in a cold gas by replacing the quantum scattering between them by classical hard-sphere scattering. Both the quantum and classical scattering events have a *cross-section* σ, which roughly measures the area of the "shadow" created by the scattering center when it is irradiated with a beam of particles. As the radius of the hard sphere increases, so too does its σ. At some critical radius a, the cross-section will match that of the quantum-mechanical scattering modeled by the sphere; this critical radius is the scattering length.

By convention, $a > 0$ describes repulsive interactions, while $a < 0$ describes attractive interactions. A system with a near zero is nearly free (non-interacting), and collisions between particles are well modeled by a contact potential. As a grows in magnitude, the system becomes strongly correlated. When a diverges, the range of the interaction is infinite, and every atom in the gas is coupled to every other; the gas is in a maximally many-body state. The interactions are the strongest allowable by quantum mechanics, limited only by the requirement of unitary time evolution, earning this situation the name *unitarity*.

2.2 Cold Fermions and Holography

The key feature of a unitary gas that distinguishes it from most others is its *scale invariance*. Normally, a finite interaction range makes an atomic gas appear nearly non-interacting when viewed from afar. For a unitary gas, however, zooming in or out does not decouple it or make interactions fade into our out of view. Since a is infinite, unitary gases are self-similar at every length scale, which is to say that they are invariant under "zooming" transformations of scale. At this point, we will restrict our discussion to fermionic gases. In the presence of attractive disturbances or potentials like those that can arise in a crystal or an optical lattice, fermions bind together to form what are called Cooper pairs; this regime is governed by the so-called BCS theory, which also attempts to provide a mechanism for superconductivity in some materials. As the temperature falls, these pairs of fermion begin to behave as individual particles with integer spin in their own right. As bosons, these cooper pairs condense into a BEC. Surprisingly, unitarity is achieved at the midpoint of this so-called BCS-BEC crossover, first observed in 2004. The unitary Fermi gas therefore lies at an important phase transition in ultracold systems; indeed, it comes with the usual hallmarks of the transition, including a divergent correlation length a, a vanishing energy gap, and scale-invariant behavior.

So far, we have been discussing fairly standard approaches to the situation. Let us therefore venture into more tentative territory by making two observations. (1) Theorists often use the words "scale-invariant" and "conformally invariant" to mean the same thing: there is a subtle technical distinction, but we will ignore it. (2) Fermions are accurately described by the formalism of quantum field theory (QFT), which builds upon and supersedes ordinary quantum mechanics. From here, we see that unitary fermions are a textbook example of an infinitely strongly coupled conformal field theory. Miraculously, this is exactly the type of QFT that possesses a dual description in terms of string theory, which attempts to give a quantum description of gravity. In other words, unitary fermions may be amenable to the techniques of this so-called holographic duality, and at the very least will reveal some interesting physics.

3 The AdS/CFT Correspondence

For well over a century, the notion of duality has prevailed as an important theme in theoretical physics. The basic idea is that two theories that appear very different in nature may actually describe the same physics. For instance, classical electrodynamics in free space is invariant under the interchange of the electric and magnetic fields, $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\frac{1}{c^2}\mathbf{E}$. Another proto-example is that of wave-particle duality in the early days of quantum mechanics. Duality is extremely powerful in its ability to reveal more information about the underlying physics than either of its two dual descriptions could give alone. For example, the Ising model, which attempts to explain the quantum origins of magnetism, is nontrivially dual to itself, with an interesting mapping between low and high temperatures. This duality reveals the existence of a critical temperature at which the Ising model exhibits a phase transition!

The duality we are most interested is complicated, so we will use the next section to explain the two theories it identifies. Crudely, the AdS/CFT correspondence states that a theory of quantum gravity forumated on a type of "bulk" spacetime called anti-de Sitter (AdS) space is equivalent to a conformal field theory (CFT) defined on the boundary of AdS. Since 1997, the AdS/CFT duality has ushered in a wealth of insights on both the gravitational and field-theoretic sides gleaned from work done on the other side.

3.1 Conformal Field Theories and Gravity

Let us, in broad strokes, paint the landscape of modern theoretical physics. We begin with classical mechanics, where the main program is to determine the trajectory of a point particle under the influence of external forces. This trajectory is determined by the *equations of motion* according to Newton, Lagrange, or Hamilton (all three are equivalent!). We then pass from classical to quantum mechanics by promoting the particle's position q and momentum p to linear operators on a Hilbert space satisfying the canonical commutation relations $[q,p] = i\hbar$. Back on the classical side, we may consider not only systems of particles, but also of continuous distributions or *fields* $\phi(x,t)$ that depend on space and time. The quantum mechanics of these fields is suitably called *quantum field theory* (QFT), and has infinitely many degrees of freedom, one for each spacetime point (x,t) at which quantum mechanics takes place.

Quantum field theories are severely restricted by the symmetries we impose upon them; we will be primarily concerned with *conformal symmetry.* Consider a *scale transformation* $(x,t) \to (\lambda x, \lambda t)$. Under this change, the fields may in special cases remain entirely invariant. We will allow something slightly worse to happen: if $\phi \to \lambda^{\Delta}\phi$ under a change of scale, then we say that the theory satisfied by ϕ is conformally invariant, i.e. a CFT. All of the above is generally true for relativistic CFTs, where space and time are treated on the same footing; for more general theories, however, we must allow t and x to scale differently: we now allow the power-law dependence $x \to \lambda x$, $t \to \lambda^z t$, with z called the *dynamical scaling exponent.* The physics at hand is governed by the free Schrödinger equation $-\frac{\hbar^2}{2m}\partial_x^2\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$, which is seen to be invariant under a $z = 2$ scaling. (λ^2 cancels out after the transformation, leaving the equation invariant.)

On the opposite end of the spectrum of physics lies general relativity (GR), which describes the universe on large scales. GR is a classical theory, and comes with an action whose variation yields the equations of motion for a dynamical object called $g_{\mu\nu}$:

$$S[g_{\mu\nu}] = \int_M \left[\frac{1}{4\pi G}(R - 2\Lambda) + \mathcal{L}_{\mathrm{M}}\right]\sqrt{-g}\,\mathrm{d}^4x \implies R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3.1}$$

The object $g_{\mu\nu}$ is called a *metric*; as a sort of drunk generalization of an inner product, it tells us how to measure distances in spacetime, and therefore describes the curvature of spacetime. The metric is determined entirely by the mass-energy content of spacetime, encoded in the object $T_{\mu\nu}$. The simplest solutions to Einstein's field equations (EFE) are those corresponding to an empty universe. These vacuum solutions must have constant spacetime curvature, since no point should have a curvature that distinguishes it from the rest. In the halls of a math department, we are told that spaces of constant curvature are either Euclidean spaces ($\mathbb{R}^n$), spheres (S^n), or hyperbolic spaces ($\mathbb{H}^n$). But this is physics: hence, recall that flat Minkowski spacetime is obtained from the Euclidean metric $\mathrm{d}s^2 = (\mathrm{d}x^0)^2 + \cdots + (\mathrm{d}x^n)^2$ by flipping the sign of the zeroth coordinate to account for the constancy of the speed of light. Similarly, flipping the sign of the time coordinate in the metrics for S^n and $\mathbb{H}^n$ yield the so-called de Sitter (dS) and anti-de Sitter (AdS) spaces.

As it turns out, AdS space will be especially important to us. This is in part because of the structure of its boundary. If an observer approaches spacelike infinity in AdS space, they will notice the spacetime start to flatten: in technical terms, we say that the $(n-1)$-dimensional *conformal boundary* of AdS_n is locally Minkowski. This is our first hint that, if a theory of gravity is formulated on an AdS background, its boundary may be a suitable environment for a relativistic conformal field theory to make its home.

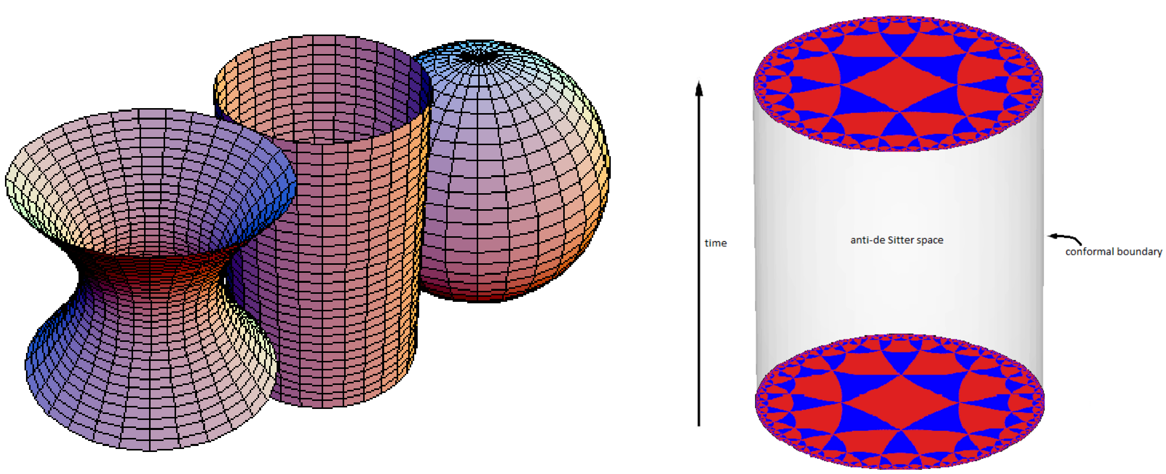

Figure 1: Spaces of constant curvature (left) and the structure of AdS space (right).

3.2 The AdS/CFT Dictionary

In 1974, Bekenstein and Hawking discovered that the entropy of a black hole scales linearly with its area. This was surprising on two counts: (1) that black holes are thermodynamic objects is unexpected, but a consequence of QFT effects in curved spacetime; and (2) this meant that there was enough information on the black hole's boundary to describe all of the states inside it. Twenty years later, Leonard Susskind elevated this statement to a fundamental principle. Given the similarities between black holes and our universe (both have horizons that nothing can escape, and singularities either in space or in time), Susskind argued that the bulk theory of our universe, a quantum theory that includes gravity, should be well described by a different theory living on the boundary of the universe, much like a hologram. This was formalized in Maldacena's 1997 paper, which claimed that the Minkowski boundary of AdS provides the spacetime for a CFT whose physics is equivalent ("holographically dual") to quantum gravity in the bulk.

Moreover, AdS/CFT is a *strong-weak duality*, meaning that the "non-stringy" semiclassical limit of string theory (i.e. GR) corresponds to strong coupling in the boundary CFT, and vice versa. This is uniquely suited to our purposes, where an infinitely strongly coupled CFT describing unitary fermions should admit a purely gravitational description in a spacetime of one dimension higher. The AdS/CFT correspondence consists mainly of a "dictionary" mapping computations in one theory to computations in the other. In particular, the generators of the symmetries constraining unitary fermions will map to isometries of the gravitational metric that describes them.

4 Holography for Galilean CFTs

Armed with the idea of holography, we will apply it to our system, a nonrelativistic CFT governed by the Schrödinger equation. Our plan of attack will be as follows. (1) We will describe the algebra of symmetries possessed by unitary fermions explicitly. (2) We will show how to embed this algebra into a conformal group in a higher dimension. (3) We will write down the AdS metric, and then deform it so as to reduce its symmetries down to those described in (1).

4.1 The Schrödinger Algebra

As mentioned earlier, the unitary Fermi gas defines an nonrelativistic CFT at infinite coupling strength. Explicitly, this theory is constrained by the *Schrödinger algebra*, which consists of the following generators: spacetime translations $P^\mu = (H, P^i)$ (i.e. the Hamiltonian and momentum operators), spatial rotations M^{ij}, Galilean boosts K^i, the $z = 2$ dilation D, a so-called special conformal transformation C, and the mass m. In addition, the unitary Fermi gas has an $SU(2)$ symmetry due to invariance under spin rotations.

We are now ready to embed this algebra into a conformal group. To do so, we begin with the massless Klein-Gordon equation in flat Minkowski space of dimension $(d+1)+1$: $\Box\psi = 0$. This is the relativistic wave equation, and is conformally invariant ($z = 1$). Next, we define light-cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{d+1})$, maximally mixing time with the $(d+1)$th spatial direction. Under this change, we have

$$\Box\psi \equiv \Big(-\partial_0^2 + \sum_{i=1}^{d+1} \partial_i^2 \Big)\psi = 0 \iff \Big(-2\frac{\partial}{\partial x^-}\frac{\partial}{\partial x^+} + \sum_{i=1}^{d} \partial_i^2 \Big)\psi = 0. \tag{4.1}$$

Now, we make the identification $\partial/\partial x^- \equiv -im$, similarly to how momentum is identified with a spatial derivative in ordinary quantum mechanics; we also let x^+ play the role of time. (In other words, identify the light-cone momenta $P^\pm$ with mass and energy, respectively.) With these modifications and after some algebra, the wave equation reads

$$\Big(2im\frac{\partial}{\partial t} + \sum_{i=1}^{d} \partial_i^2 \Big)\psi = 0 \iff i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi. \tag{4.2}$$

Thus the Schrödinger equation in d spatial dimensions appears out of a relativistic wave equation in $d+1$ spatial dimensions. This, in turn, shows that the Schrödinger algebra Sch(d) may be embedded into the conformal algebra $O(d+2, 2)$. This group also turns out to be the symmetry group of AdS space.

4.2 The Dual Spacetime

Recall that an infinitely strongly coupled CFT should admit a dual description at the level of pure general relativity. On the CFT side, we have showed that Sch(d) embeds into $O(d+2,2)$. Let us therefore move up another dimension—into the bulk. Beginning with the AdS metric in *Poincaré coordinates* (x^0, x^i, r), we will again introduce light-cone coordinates (x^+, x^i, x^-, r) and tweak the metric slightly:

$$\mathrm{d}s^2 = \frac{1}{r^2}\Big(\eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}r^2\Big) = \frac{1}{r^2}\Big(-\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + \mathrm{d}r^2\Big) \longrightarrow \tag{4.3}$$

$$\frac{1}{r^2}\Big(-\frac{2\,\mathrm{d}x^{+2}}{r^2} - 2\,\mathrm{d}x^-\mathrm{d}x^+ + \mathrm{d}\mathbf{x}^2 + \mathrm{d}r^2\Big) = -\frac{2\,\mathrm{d}t^2}{r^4} + \frac{(i/m)\mathrm{d}t + \mathrm{d}\mathbf{x}^2 + \mathrm{d}r^2}{r^2} = \mathrm{d}s_*^2. \tag{4.4}$$

It can be shown, as required by the AdS/CFT dictionary, that the generators of the Schrödinger algebra are all isometries of the metric $\mathrm{d}s_*^2$. For instance, the $z = 2$ scaling symmetry is manifest in the powers of r appearing in the denominator: time is scaled twice as much as space.

Unfortunately, not much else is understood about this spacetime. Nevertheless, it is possible to prove the following. Descended from AdS space, it has negative curvature caused by the *cosmological constant* $\Lambda < 0$ appearing in the EFE. Moroever, this spacetime is a solution of the EFE for a universe filled with a uniform pressureless dust; this ensures the asymptotically nonrelativistic behavior of the metric. Furthermore, since the mass spectrum of unitary fermions is discrete, it is possible that the x^- dimension responsible for mass is compactified á la Kaluza-Klein; the compactification radius would be set by the mass of the fermions.

5 Summary and Outlook

The analysis presented here is admittedly both complicated and tentative, yielding no concrete results or immediate applications. Nevertheless, it is an interesting attempt to synthesize disparate areas of physics with strictly nonnegative success. We started by considering a gas of ultracold fermions. Taking correlations of infinite length $a \to \infty$, we found the system at a phase transition, where it exhibited universal, scale-invariant behavior at $z = 2$. We then viewed the unitary Fermi gas as a strongly coupled conformal field theory and argued that it should have a dual, purely gravitational, description in a higher-dimensional space. We achieved this duality by embedding the algebra of Schrödinger symmetries governing fermions at unitarity into a larger conformal ($z = 1$) group; we then deformed the corresponding anti-de Sitter metric to take its symmetries from the full conformal group down to those of the embedded Schrödinger algebra. This metric describes a universe with a large extra dimension and a compactified extra dimension representing mass, negatively curved in the presence of vacuum energy, and filled with dust. The main takeaway, above these results, is that physics is awesome.

As a closing remark, it should be noted that this type of analysis is not the first of its kind. Condensed-matter physics has a rich history of using tools from high-energy theory in surprising ways. Even the AdS/CFT correspondence itself has proved useful, in particular when dealing with superfluids. Several connections between superfluids and black holes have been discovered, telling us that perhaps the deep connections between disparate fields are more nontrivial and surprising than we thought.